Section 14.4

Differentiability and Tangent Planes

1 Geometry of Tangent Planes

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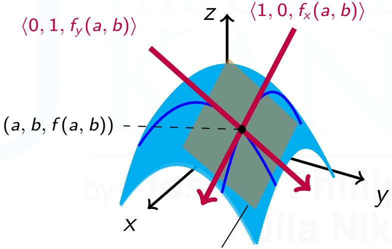
Tangent Planes

Let $P = (a, b, c)$ be a point on the surface S defined by $z = f(x, y)$.

The **tangent plane** to the surface S at the point P is the plane containing tangent lines to *every* curve on S passing through P.



Tangent Plane to surface of $z = f(x, y)$ at $(a, b, f(a, b))$.

The tangent plane is the **best linear approximation** to $z = f(x, y)$ near P (just like the tangent line to the graph of a function $y = f(x)$).

2 Equations for Tangent Planes

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Equations for Tangent Planes

The tangent plane to the graph of $f(x, y)$ at the point $P(a, b, c)$ contains the two tangent lines through P with direction vectors

$$\langle 1, 0, f_x(a, b)\rangle, \qquad \langle 0, 1, f_y(a, b)\rangle.$$

Normal vector to the tangent plane:

$$\begin{aligned}\vec{n} &= \langle 1,\ 0,\ f_x(a, b)\rangle \times \langle 0,\ 1,\ f_y(a, b)\rangle \\ &= \langle -f_x(a, b),\ -f_y(a, b),\ 1\rangle\end{aligned}$$

Equation of the tangent plane:

$$\vec{n} \cdot \langle x - a,\ y - b,\ z - f(a, b)\rangle = 0$$

$$-f_x(a, b)(x - a) - f_y(a, b)(y - b) + (z - f(a, b)) = 0$$

$$\boxed{z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)}$$

Tangent Planes

Example 1: Let $f(x, y) = x^3 + y^2$. Find the equation of the tangent planes to the graph of f at $P(1, 1, 2)$ and $Q(-1, 2, 3)$.

Solution:

$$f_x(x, y) = 3x^2 \qquad f_y(x, y) = 2y$$
$$f_x(1, 1) = 3 \qquad f_y(1, 1) = 2$$
$$f_x(-1, 2) = 3 \qquad f_y(-1, 2) = 4$$

Tangent plane at P: $z = 2 + 3(x - 1) + 2(y - 1)$
Tangent plane at Q: $z = 3 + 3(x + 1) + 4(y - 2)$

Warning: The equation of the tangent plane at (a, b) is *not*

$$z = f(a, b) + 3x^2(x - a) + 2y(y - b) \text{ !!}$$

3 Differentiability

Tangent Planes: Warning

Warning: There are some weird functions out there such that

- $f(x, y)$ is continuous at $(x, y) = (a, b)$;
- $f_x(a, b)$ and $f_y(a, b)$ exist; but
- ... the plane $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$ is not tangent to the graph of f.

We'll see an example of this soon (Example 4).

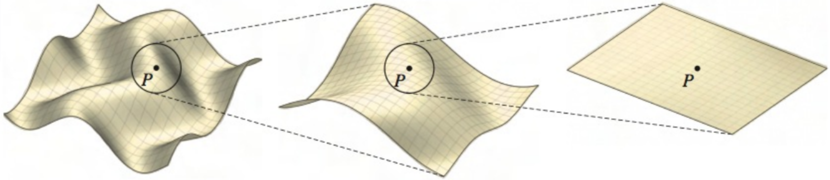
This weird behavior does not happen for functions of one variable!

Fortunately, it does not happen for elementary functions (polynomials, rational functions, trig/log/exponential functions).

Differentiability

Recall that a one-variable function $y = f(x)$ is **differentiable** at $x = a$ if the derivative $f'(a)$ exists, which means that the graph can be approximated near $x = a$ by a line (the tangent line).

A two-variable function $f(x, y)$ is **differentiable** at (a, b) if it is **locally linear**.

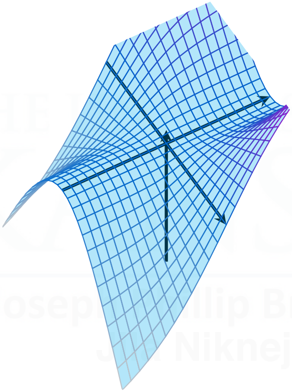


That is, the graph can be approximated near $(x, y) = (a, b)$ by a linear function (the tangent plane). The closer you zoom in on the graph, the closer it gets to a plane.

Differentiability and Non-Differentiability

Example 2:

$$z = \begin{cases} \frac{x^2 y}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$



$-1 \leq x, y \leq 1$

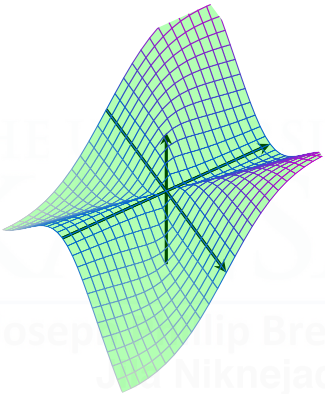
Link

This function is **differentiable**: the more you zoom in, the more the graph resembles a plane.

Differentiability and Non-Differentiability

Example 3:

$$z = \begin{cases} \frac{x^2 y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$



$-1 \leq x, y \leq 1$

Link

This function is **not differentiable**: the graph does not flatten out upon zooming in.

Algebraic Definition of Differentiability

Suppose that $(x, y) = (a, b)$ is in the domain of a function $z = f(x, y)$.

We know that the tangent plane, <u>if it exists,</u> has the equation

$$L_{(a,b)}(x, y) = z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b).$$

The function is **differentiable** at $(x, y) = (a, b)$ if the linear function $L_{a,b}$ is a better and better approximation to the graph of f as $(x, y) \to (a, b)$:

$$\lim_{(x,y)\to(a,b)} \frac{L_{(a,b)}(x, y) - f(x, y)}{\sqrt{(x - a)^2 + (y - b)^2}} = 0.$$

However, this criterion is awkward to check

Differentiability vs. Partial Derivatives

A single-variable function $f(x)$ is differentiable at any number a where $f'(a)$ exists.

However, it is not true that a two-variable function $f(x, y)$ is differentiable wherever $f_x(a, b)$ and $f_y(a, b)$ exist.

We need a stronger criterion:

Criterion for Differentiability

If **both partial derivatives** f_x and f_y are **continuous** at (a, b), then $f(x, y)$ is **differentiable** at (a, b).

In particular, all elementary functions (such as polynomials, rational functions, trig/log/exponential functions) are differentiable on their domains. (But be careful with piecewise functions!)

Differentiability vs. Partial Derivatives

Example 4: Let's explain further the properties of the function in Example 3. Let $f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{for } (x,y) \neq (0,0), \\ 0 & \text{for } (x,y) = (0,0). \end{cases}$

This function is continuous. Moreover,

$$f_x(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x \to 0} \frac{0-0}{x} = 0,$$
$$f_y(0,0) = \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y} = \lim_{y \to 0} \frac{0-0}{y} = 0.$$

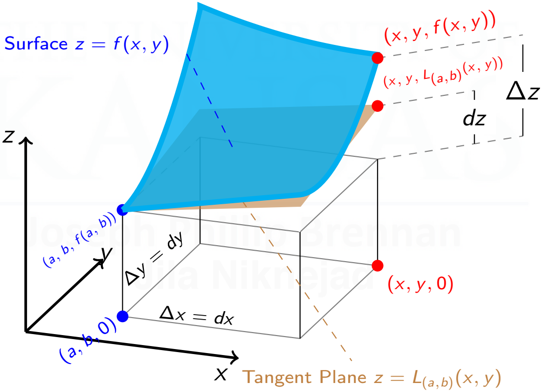
But $f(x,y)$ is not differentiable at $(0,0)$ (see Example 3). While f_x and f_y **exist** at $(0,0)$, they are **not continuous** at $(0,0)$. For example,

$$\lim_{(x,y)\to(0,0)} f_y(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2(x^2-y^2)}{(x^2+y^2)^2} \ \text{ DNE.}$$

4 Linear Approximation and Total Differentials

Approximation with Tangent Planes

- Blue surface: graph of $z = f(x, y)$
- Yellow plane: tangent plane to graph at $(a,\ b,\ f(a, b))$
- $\Delta x = dx$, $\Delta y = dy$: small change in x, y values from a, b.
- Δz = **actual** change in z from (a, b) to $(a + \Delta x, b + \Delta y)$.
- dz = **estimated** change in z using tangent plane (**total differential**).



Linear Approximation

If f is differentiable, then the tangent plane $z = L_{(a,b)}(x, y)$ is the best linear approximation to $z = f(x, y)$ near (a, b).

That is, $f(x, y) \approx L_{(a,b)}(x, y)$ when $\sqrt{(x-a)^2 + (y-b)^2}$ is small.

To put it another way:

$$\begin{aligned}\Delta z \approx dz &= f_x(a, b)(x-a) + f_y(a, b)(y-b) \\ &= f_x(a, b)\Delta x + f_y(a, b)\Delta y\end{aligned}$$

To put it another way:

$$\begin{aligned}f(x, y) \approx L_{(a,b)}(x, y) &= f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b) \\ &= f(a, b) + \underbrace{f_x(a, b)\Delta x + f_y(a, b)\Delta y}_{dz}\end{aligned}$$

Linear Approximation

Example 5: Show that $f(x, y) = y^2 e^{xy}$ is differentiable at $(0, 2)$. Find the linear approximation of f at $(0, 2)$ and use it to approximate $f(0.1, 1.9)$.

<u>Solution:</u> Using the Criterion for Differentiability:

$$f_x(x, y) = y^3 e^{xy} \qquad\qquad f_y(x, y) = y(2 + xy)e^{xy}$$

Since f_x and f_y are continuous everywhere, f is differentiable everywhere.

Tangent plane at $(0, 2, 4)$:

$$\begin{aligned} L_{(0,2)}(x, y) &= f_x(0, 2)(x - 0) + f_y(0, 2)(y - 2) + 4 \\ &= 8x + 4(y - 2) + 4 = 8x + 4y - 4 \end{aligned}$$

Approximation:

$$f(0.1, 1.9) \approx L_{(0,2)}(0.1, 1.9) = 4 + 8(0.1) + 4(1.9 - 2) = \boxed{4.4}.$$

(Actual value: $f(0.1, 1.9) \approx 4.3654$.)

Linear Approximation

Example 6: Use a linear approximation to estimate $(3.1)^2(1.9)^3$.

Solution: Let $f(x, y) = x^2y^3$. Then f is differentiable everywhere. To approximate $f(3.1,\ 1.9)$, find the tangent plane approximation near $(3, 2)$.

$$\begin{array}{lll} & f_x(x, y) = 2xy^3 & f_y(x, y) = 3x^2y^2 \\ f(3, 2) = 72 & f_x(3, 2) = 48 & f_y(3, 2) = 108 \end{array}$$

Since $(3.1,\ 1.9)$ is near $(3, 2)$ with $\Delta x = .1$ and $\Delta y = -.1$,

$$f(3.1,\ 1.9) \approx L_{(3,2)}(3.1,\ 1.9) = 48(.1) + 108(-.1) + 72 = \boxed{66.}$$

(Exact value: $(3.1)^2(1.9)^3 = 65.914\ldots$)

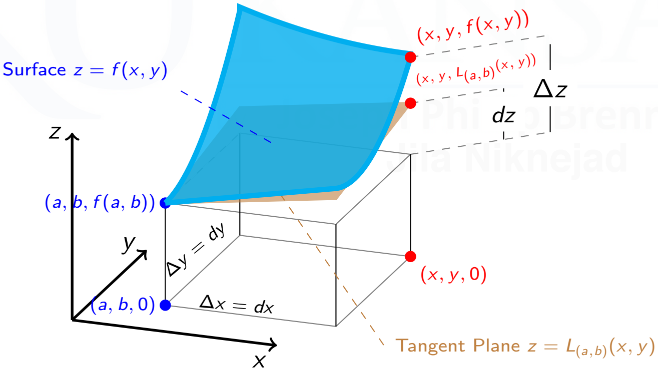
Total Differentials

The **total differential** of $z = f(x, y)$ is

$$dz = f_x(x, y)dx + f_y(x, y)dy = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y$$

The total differential can be used to approximate the **increment**

$$\Delta z = f(a + \Delta x,\ b + \Delta y) - f(a, b).$$



Application of Total Differentials

Example 7: Use the total differential to estimate the amount of metal in a closed cylindrical can with radius 4 cm and height 12 cm if the sides are 0.04 cm thick and the top and bottom are 0.01 cm thick.

$$\underline{\text{Answer}} = \text{volume of cylinder outside can} - \text{volume of cylinder inside can}$$
$$= V(4, 12) - V(3.96, 11.98)$$

where $V(r, h) = \pi r^2 h$. We want to calculate $|\Delta V|$, where

$$\Delta V \approx dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh.$$

Substitute in

$$\begin{array}{cccc} r = 4 & V_r(r, h) = 2\pi rh & h = 12 & V_h(r, h) = \pi r^2 \\ \Delta r = -0.04 & V_r(4, 12) = 96\pi & \Delta h = -0.02 & V_h(4, 12) = 16\pi \end{array}$$

so

$$|dV| = 96\pi(0.04) + 16\pi(0.02) = 4.16\pi \, \text{cm}^3 \approx 13.069 \, \text{cm}^3.$$

Section 14.4 Summary

Section 14.4 Summary for after Section 14.5 is completed.